SOLVING HEAT-CONDUCTION EQUATION FOR A TWO-COMPONENT CYLINDER AND THERMAL DESIGN OF A dc MOTOR

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UDC 536.24

The solution is obtained of the heat-conduction equation for a two-component cylinder, and results are presented of its application to thermal design of a dc motor.

The solution of the heat-conduction equation for a two-component cylinder has numerous applications. In particular, it can be applied to the important problem of thermal design of a dc motor if the teeth of the armature and slots with conductors are replaced by an equivalent ring-like continuous component.*

Such design enables one to link the admissible load factor of the motor with the duration of the motor being on; consequently, one should be able to lower the weight and to reduce the clearances of the control. The computations, the results of which are given below, were carried out for a wide range of values of dimensionless parameters encountered in thermal design of a dc motor. The outline of the computations is shown in Fig. 1.

A dimensionless coordinate x is now introduced which expresses the distance to the axis in terms of the outer radius of the ring the following notation being introduced:

$$\frac{r}{R}=\varepsilon,\quad \frac{1}{R^2}\frac{\Lambda}{C}=a,\quad \frac{1}{R^2}\frac{\lambda}{c}=b,$$

$$\frac{q_0}{C}=w_0,\quad \frac{q}{c}=w,\quad \frac{\lambda}{\Lambda}=\mu,$$

$$\frac{b}{a}=\beta\mu,\quad \frac{R\alpha}{\Lambda}=\delta,\quad \frac{\delta}{\mu}=H.$$

For the temperature distribution in the rod $\theta(x, t)$ and in the ring $\vartheta(x, t)$ one now obtains the following system of heat-conduction equations and the following boundary conditions:

$$\left.\begin{aligned}\frac{\partial\theta}{\partial t}&=a\left(\frac{\partial^2\theta}{\partial x^2}+\frac{1}{x}\frac{\partial\theta}{\partial x}\right)+w_0\\ \frac{\partial\vartheta}{\partial t}&=b\left(\frac{\partial^2\vartheta}{\partial x^2}+\frac{1}{x}\frac{\partial\vartheta}{\partial x}\right)+w,\end{aligned}\right\}\tag{1}$$

$$\frac{\partial\theta(0,\ t)}{\partial x}=0,$$

*This simplification is widely applied in thermal design (see, for example [1]).

Leningrad Institute of Mechanics. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 27, No. 1, pp. 151-158, July, 1974. Original article submitted August 6, 1973.

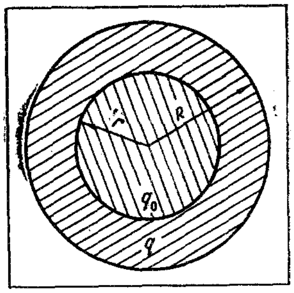

Fig. 1. The calculation procedure.

$$\theta(\varepsilon, t) = \vartheta(\varepsilon, t)$$

$$\frac{\partial\theta(\varepsilon, t)}{\partial x} = \mu\frac{\partial\vartheta(\varepsilon, t)}{\partial x}, \tag{2}$$

$$\frac{\partial\vartheta(1, t)}{\partial x} + H\vartheta(1, t) = 0,$$

$$\left.\begin{aligned}\theta(x, 0) &= F(x),\\ \vartheta(x, 0) &= f(x).\end{aligned}\right\} \tag{3}$$

A solution of the corresponding homogeneous system is given by

$$\left.\begin{aligned}\theta_0(x, t) &= \Sigma A_n J_0(k_n x)\exp(-k_n^2 at),\\ \vartheta_0(x, t) &= \Sigma A_n[g_n J_0(l_n x) + h_n Y_0(l_n x)]\exp(-l_n^2 bt),\end{aligned}\right\} \tag{4}$$

where the characteristic values k_n and l_n are solutions of the system of equations

$$l_n^2 b = k_n^2 a,$$

$$\begin{vmatrix} J_0(l_n\varepsilon) & Y_0(l_n\varepsilon) & J_0(k_n\varepsilon)\\ J_1(l_n\varepsilon) & Y_1(l_n\varepsilon) & \sqrt{\frac{\beta}{\mu}}J_1(k_n\varepsilon)\\ HJ_0(l_n) - l_n J_1(l_n) & HY_0(l_n) - l_n Y_1(l_n) & 0\end{vmatrix} = 0,$$

and subsequently the coefficients g_n and h_n are computed by using the formulas

$$\left.\begin{aligned} g_n &= \frac{\sqrt{\mu}J_0(k_n\varepsilon)Y_1(l_n\varepsilon) - \sqrt{\beta}J_1(k_n\varepsilon)Y_0(l_n\varepsilon)}{\sqrt{\mu}[J_0(l_n\varepsilon)Y_1(l_n\varepsilon) - J_1(l_n\varepsilon)Y_0(k_n\varepsilon)]},\\ h_n &= \frac{\sqrt{\beta}J_0(l_n\varepsilon)J_1(k_n\varepsilon) - \sqrt{\mu}J_0(k_n\varepsilon)J_1(l_n\varepsilon)}{\sqrt{\mu}[J_0(l_n\varepsilon)Y_1(l_n\varepsilon) - J_1(l_n\varepsilon)Y_0(l_n\varepsilon)]}.\end{aligned}\right\} \tag{5}$$

The coefficients A_n are determined so that the obtained solution satisfies the given initial conditions:

$$A_n = \frac{1}{G_n}\left\{\int_0^\varepsilon xF(x)J_0(k_n x)\,dx + \frac{1}{\beta}\int_\varepsilon^1 xf(x)[g_n J_0(l_n x) + h_n Y_0(l_n x)]\,dx\right\},$$

where

$$G_n = \frac{\varepsilon^2}{2}\left[\left(1-\frac{1}{\mu}\right)J_1^2(k_n\varepsilon) + \left(1-\frac{1}{\beta}\right)J_0^2(k_n\varepsilon)\right] + \left(\frac{\mu H^2}{2k_n^2} - \frac{1}{2\beta}\right)[g_n J_0(l_n) + h_n Y_0(l_n)]^2. \tag{6}$$

For dc motors the values of the dimensionless parameters may be found in the following ranges: $\beta = 1$–1.07; $\varepsilon = 0.5$–0.85; $\delta = 0.15$–0.4; $\mu = 0.3$–3.0.

In calculating the characteristic values k_n the mean value of $\beta = 1.04$ was adopted.

The characteristic values increase sufficiently rapidly so that the corresponding terms of the series become negligibly small already for a small t compared with the first term. Actual calculations carried out when testing experimentally the procedure have shown that already for t = 0.6 min there is the ratio

$$\exp(-k_3^2 at) : \exp(-k_1^2 at) \approx 0{,}1.$$

Therefore, for the range of parameter values (ε, δ, μ) under consideration it is sufficient to have available the values of k_1 and k_2 which are represented on the nomogram (Fig. 2).

In designing a motor the case in which the power of heat sources is kept constant is the most important one in practice (steel and copper losses).

The solution of the inhomogeneous system is

$$\theta_1(x) = \theta_1(0) - \frac{w_0}{4a}x^2,$$

$$\vartheta_1(x) = -\frac{w}{4b}x^2 + M\ln x + N, \tag{7}$$

where

$$M = \frac{\varepsilon^2}{2}\left(\frac{w}{b} - \frac{w_0}{\mu a}\right);$$

$$N = \frac{w}{4b} + \frac{1}{H}\left(\frac{w}{2b} - M\right);$$

$$\theta_1(0) = \frac{\varepsilon^2}{4}\left(\frac{w_0}{a} - \frac{w}{b}\right) + M\ln\varepsilon + N,$$

and in this case a steady temperature field is described. With zero initial conditions the following expression is obtained for the coefficients A_n:

$$A_n = \frac{1}{G_n}\left[\theta_1(0)B_n^1 - \frac{w_0}{a}B_n^2 + NB_n^3 - \frac{w}{b}(B_n^4 + B_n^5) + M(B_n^6 + B_n^7\ln\varepsilon) - NB_n^7\right], \tag{8}$$

in which

$$B_n^1 = \frac{\varepsilon}{k_n}J_1(k_n\varepsilon);$$

$$B_n^2 = \frac{\varepsilon^2}{4k_n^2}\left[2J_0(k_n\varepsilon) + \left(k_n\varepsilon - \frac{4}{k_n\varepsilon}\right)J_1(k_n\varepsilon)\right];$$

$$B_n^3 = \frac{1}{\beta l_n}[g_nJ_1(l_n) + h_nY_1(l_n)];$$

$$B_n^4 = \frac{g_n}{4\beta l_n}\left\{\frac{2}{l_n}J_0(l_n) - \left(\frac{4}{l_n^2} - 1\right)J_1(l_n) - \varepsilon^3\left[\frac{2}{l_n\varepsilon}J_0(l_n\varepsilon) - \left(\frac{4}{l_n^2\varepsilon^2} - 1\right)J_1(l_n\varepsilon)\right]\right\};$$

$$B_n^5 = \frac{h_n}{4\beta l_n}\left\{\frac{2}{l_n}Y_0(l_n) - \left(\frac{4}{l_n^2} - 1\right)Y_1(l_n) - \varepsilon^3\left[\frac{2}{l_n\varepsilon}Y_0(l_n\varepsilon) - \left(\frac{4}{l_n^2\varepsilon^2} - 1\right)Y_1(l_n\varepsilon)\right]\right\};$$

$$B_n^6 = \frac{1}{\beta l_n^2}\{g_n[J_0(l_n) - J_0(l_n\varepsilon)] + h_n[Y_0(l_n) - Y_0(l_n\varepsilon)]\};$$

$$B_n^7 = \frac{\varepsilon}{\beta l_n}[g_nJ_1(l_n\varepsilon) + h_nY_1(l_n\varepsilon)].$$

The obtained solution was used for thermal design of a dc motor. The result was experimentally tested on dc motor MI-41T with 1.6 kW power and nominal speed of 2500 rpm. The temperature of armature winding was measured on a specially constructed experimental stand and heating time was found to bring it to the temperature of 105°C for various constant loadings. The heat-exchange coefficient $\alpha = 140$ W/$m^2\cdot$deg and the equivalent heat-conduction coefficient of the ring $\lambda = 26$ W/m·deg were calculated for steady temperature which was measured for several loads.

As an example of using such nomograms it will be shown how to determine the characteristic values k_1 and k_2 for a two-component cylinder, that is, an evaluation procedure for motor armature. The original data: α and λ were given above; $R = 6.75\cdot10^{-2}$ m; $r = 4.4\cdot10^{-2}$ m; $\Lambda = 43$ W/m·deg. One now finds the following:

$$\varepsilon = \frac{4.4\cdot10^{-2}}{6.75\cdot10^{-2}} = 0.65;$$

$$\delta = \frac{6.75\cdot10^{-2}\cdot140}{43} = 0.22;$$

$$\mu = \frac{26}{43} = 0.61.$$

On the nomogram one now finds the families of curves $\mu(k_1)$ and $\mu(k_2)$ which correspond to $\varepsilon = 0.65$ and one draws the straight line $\delta = 0.22$ until it intersects the curve $\mu(k_1) = 0.6$ (the nearest value of μ). By dropping a perpendicular on the k_1 scale one reads off $k_1 = 0.64$. To find the value of k_2 we interpolate in μ:

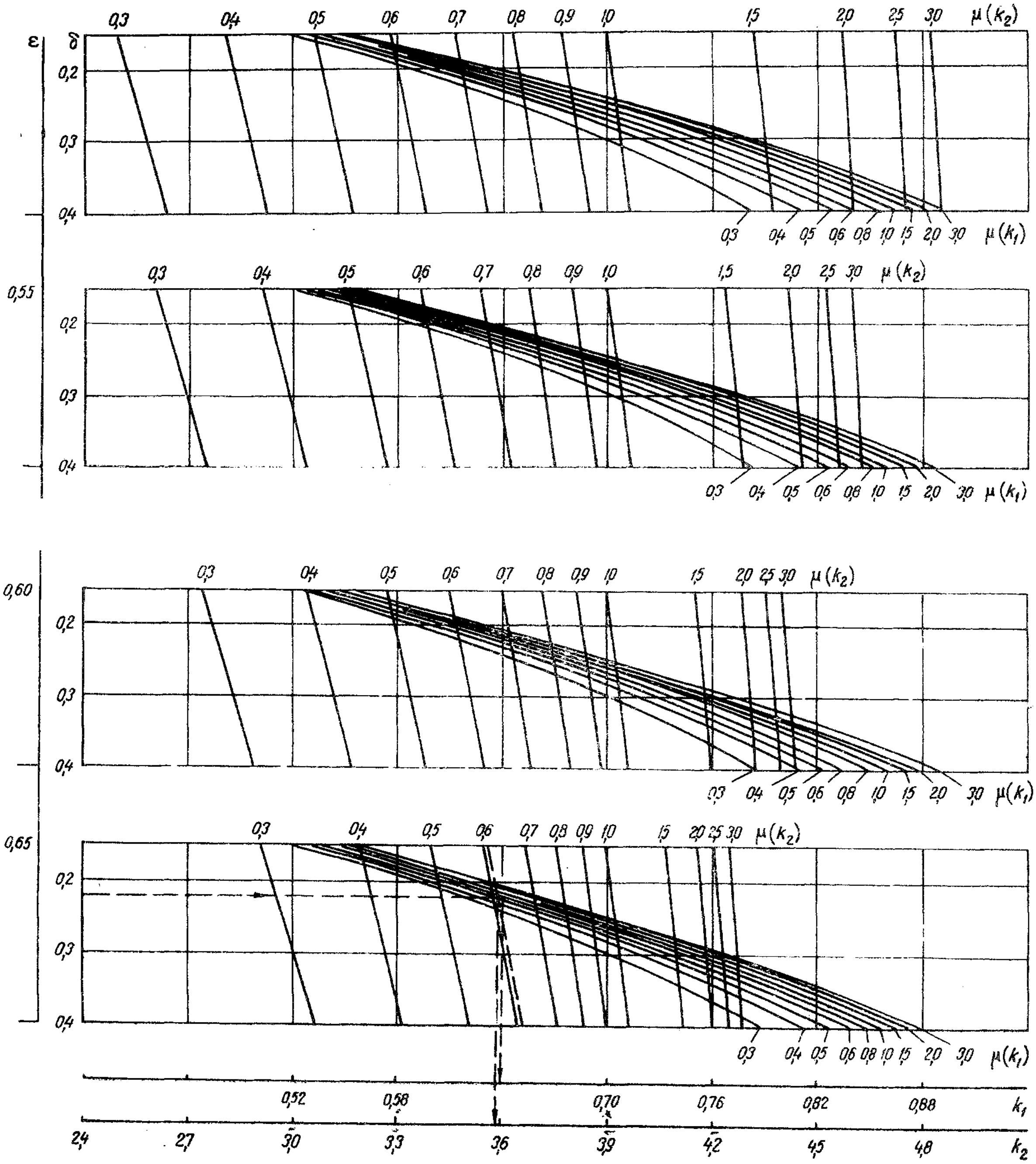

Fig. 2a. Nomogram to determine characteristic values (ε = 0.50-0.65).

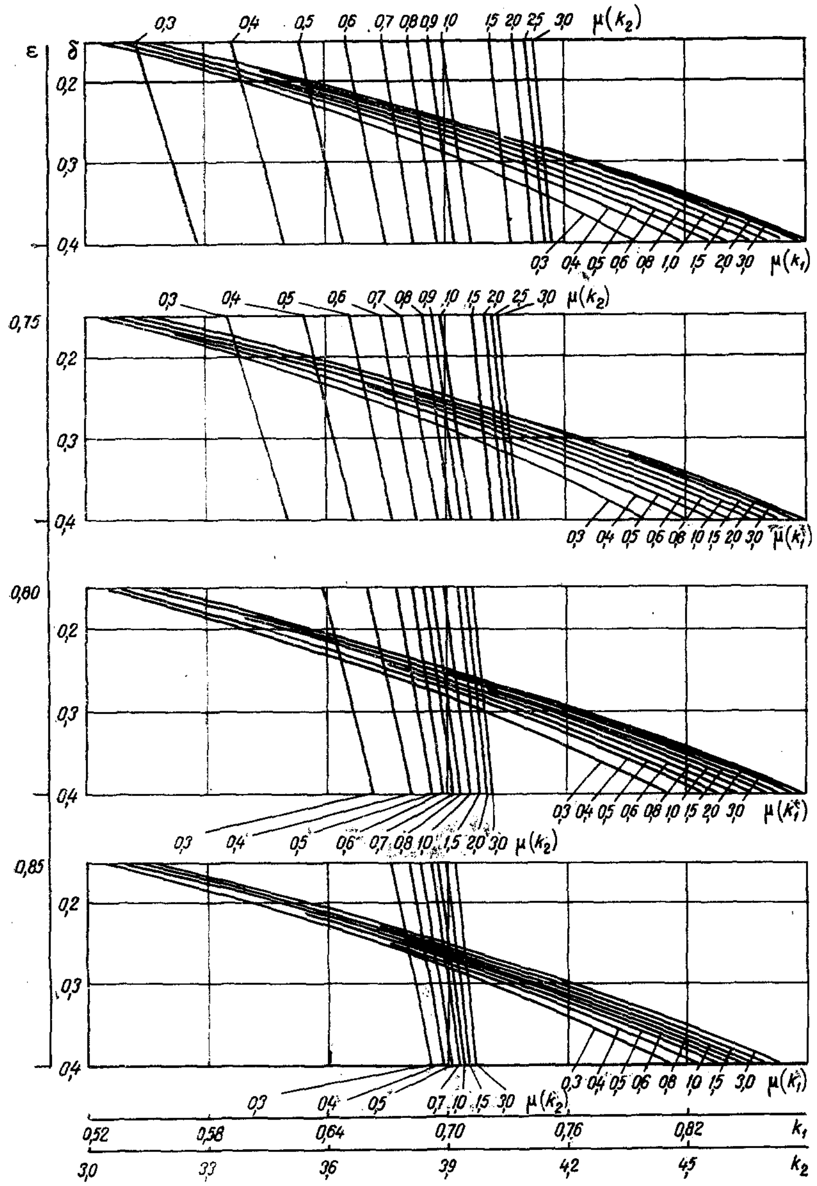

Fig. 2b. Nomogram to determine characteristic values (ε = 0.70–0.85).

between the curves $\mu(k_2) = 0.6$ and $\mu(k_2) = 0.7$ the curve $\mu(k_2) = 0.61$ is drawn. Dropping the perpendicular from the intersection point with the straight line $\delta = 0.22$ on the k_2 scale one reads off the value $k_2 = 3.59$.

The described procedure is shown by dashed lines on the nomogram.

In thermal design of a motor one must also calculate the parameter a, give the coordinate of the estimated point, and the load and calculate the terms of the series (4) by using the formulas (5)-(8).

For steel $C = 377 \cdot 10^4$ J/m$^3 \cdot$deg, hence

$$a = \frac{43}{377 \cdot 10^4 \cdot 6.75^2 \cdot 10^{-4}} = 0.0025 \text{ sec}^{-1} = 0.150 \text{ min}^{-1};$$

$$k_1^2 a = 0.64^2 \cdot 0.150 = 0.061 \text{ min}^{-1};$$

$$k_2^2 a = 3.59^2 \cdot 0.150 = 1.94 \text{ min}^{-1}.$$

The heating of the engine up to the permissible temperature – even with considerable overloading – takes more than 5 min; it, therefore, suffices to consider only the first terms in the series (4).

Knowing the value of B_1^1–B_1^7, g_1, h_1, G_1 one can compute all the coefficients k_1, then for any given constant loading one finds the coefficient A_1 and by using the formulas (4) one can finally calculate the temperature as a function of time at any point of the armature.

We omit the calculations and only note that for overloads in the current from 1.5 to 4.0 the heating time for the temperature to reach 105°C exceeded that evaluated for the hottest point ($x = 0.9$); their ratio was within the limits 1.3-1.1. This is a considerable improvement on the result obtainable by using the classical heating theory of electric motors [2], and may be considered as fully satisfactory in many applications in practice.

NOTATION

R	is the outer radius of ring;
r	is the inner radius of ring equal to the radius of the rod;
Λ, λ	are the heat conductivities of the material of rod and ring;
C, c	are the heat capacities of unit volume of rod and ring;
α	is the heat-exchange coefficient of outer rod surface;
q_0, q	are the power of uniformly distributed heat sources in rod and ring;
J, Y	are the Bessel functions of the first and second kind.

LITERATURE CITED

1. P. Schneider, Engineering Heat-Conduction Problems [Russian translation], IL, Moscow (1960).
2. G. Gotter, Heating and Cooling of Electrical Machines [Russian translation], Gosenergoizdat, Moscow (1961).